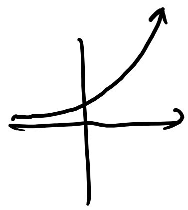
Lecture 12

Friday, September 30, 2016 8:49 AM

3.11 Hyperbolic Functions

- Certain combinations of e^x and e^{-x} arise frequently in mathematics and are given special names.

- Hyperbolic functions ✓



DEFN

1) $\underbrace{\sinh}_{\text{Hyperbolic sine}} x = \dfrac{e^x - e^{-x}}{2}$

2) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

3) $\tanh x = \dfrac{\sinh x}{\cosh x}$

4) $\operatorname{csch} x = \dfrac{1}{\sinh x}$

5) $\operatorname{sech} x = \dfrac{1}{\cosh x}$

6) $\coth x = \dfrac{1}{\tanh x}$

Identities

a) $\cosh^2 x - \sinh^2 x = 1$

Pf $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{\qquad}{2}\right)$

$$= \frac{e^{2x} + 2e^x \cdot e^{-x} + e^{-2x}}{4} - \left(\frac{e^{2x} - 2e^x \cdot e^{-x} + e^{-2x}}{4}\right)$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right)$$

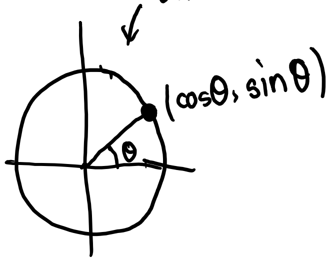
$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4} = \frac{4}{4} = 1$$

4

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

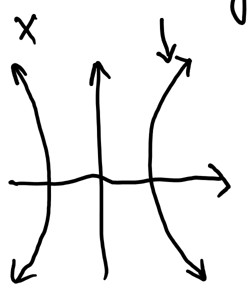
<u>Q</u> Why are they called hyperbolic funcs ?

unit circle $x^2+y^2 = 1$



$(\cos\theta, \sin\theta)$ lies on the circle because
$\cos^2\theta + \sin^2\theta = 1$

Now consider the hyperbola $x^2 - y^2 = 1$



Then any point $(\cosh t, \sinh t)$
lies on the graph (right branch)
because $\cosh^2 t - \sinh^2 t = 1$

<u>Rmk</u> Unfortunately t is not the angle.

b) $\cosh(-x) = \cosh x \rightarrow$ "Even function"

c) $\sinh(-x) = -\sinh x \rightarrow$ "Odd function"

DIY

d) $1 - \tanh^2 x = \operatorname{sech}^2 x$

<u>Derivatives</u>

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left[\frac{e^{x}-e^{-x}}{2}\right] = \frac{1}{2}\frac{d}{dx}\left[e^{x}-e^{-x}\right]$$

$$= \frac{1}{2}\left[e^{x}-e^{-x}\cdot(-1)\right] = \frac{1}{2}\left[e^{x}+e^{-x}\right]$$

chain rule

$$= \cosh x$$

Similarly, $\frac{d}{dx}[\cosh x] = \sinh x$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

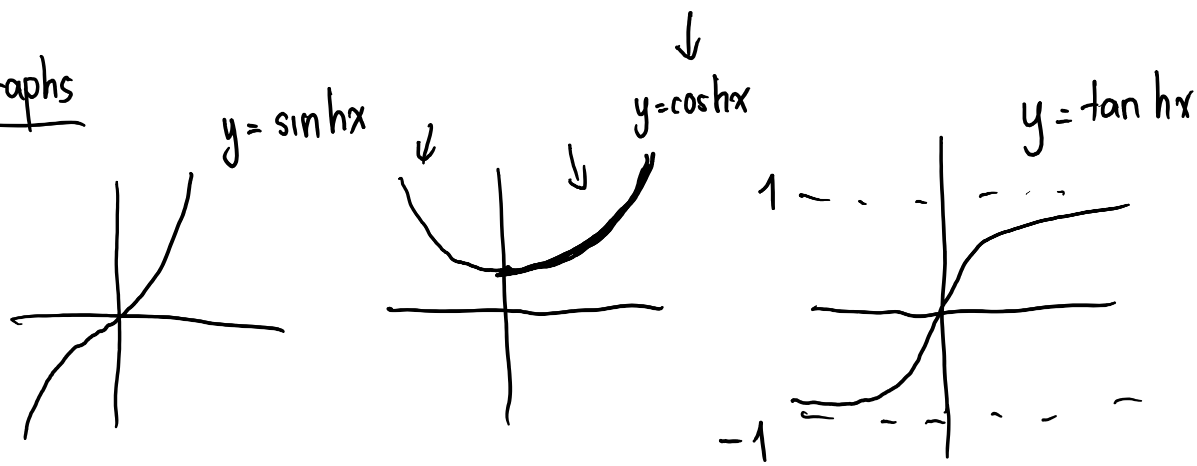
$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \cdot \coth x$$

DIY

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Graphs



sinh, tanh are 1-1 functions, cosh becomes 1-1 when restricted to $[0,\infty)$

Inverse hyperbolic functions

$$y = \sinh^{-1} x \iff \sinh y = x$$

$$y = \cosh^{-1} x \iff x = \cosh y \ , \ y \geq 0$$

$$y = \tanh^{-1} x \iff x = \tanh y$$

$$\downarrow$$

- DIY

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \ , \ x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \ , \ x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \ , \ -1 < x < 1$$

Derivatives of inverse hyperbolic functions

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}} \checkmark$$

$y = \sinh^{-1} x$, want to find y'

$\Downarrow \quad \downarrow$

$\sinh y = x$

$\Downarrow$ I.D

$\cosh y \cdot y' = 1 \Rightarrow y' = \dfrac{1}{\cosh y} = \dfrac{1}{\sqrt{1+x^2}}$

$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \underline{\cosh}^2 y = 1 + \sinh^2 y = 1 + x^2$

$\Rightarrow \cosh y = \sqrt{1+x^2}$ (since $\cosh > 0$)

<u>DIY</u>

- $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2 - 1}}$

$\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1 - x^2}$

<u>4.1</u> Maximum and minimum values

Extreme values :

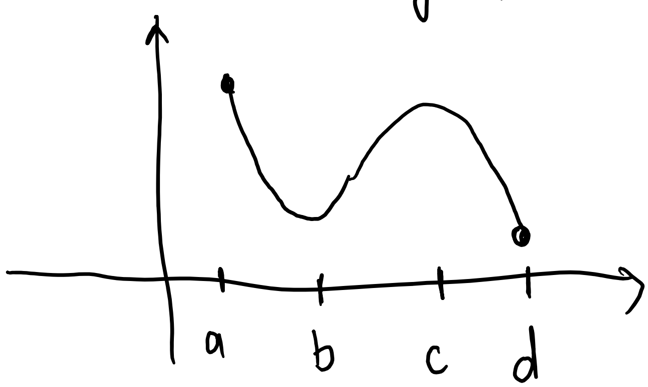
DEFN Let c be in the domain of f (D). Then

a) f(c) is the absolute max of f on D if $f(c) \geq f(x)$ for all x in D.

b) f(c) is the absolute min " " " " " $f(c) \leq f(x)$ " " " ".

$y = f(x)$

Ex



Abs max f(a)

Abs min f(d)

• DEFN

The number f(c) is called

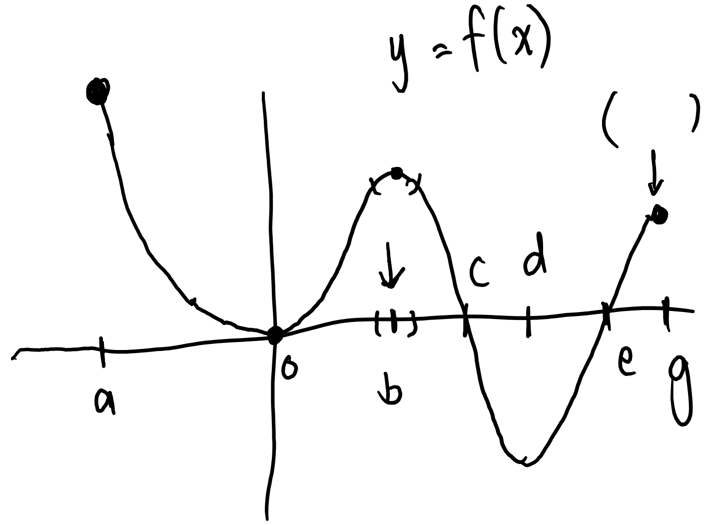
1) a local max value of f if $f(c) \geq f(x)$ when x is near c.

2) a local min " " " " $f(c) \leq f(x)$ " " " ".

RmK near c means some open interval containing c.

$y = f(x)$

Ex



$f(b) \geq f(x)$ for all x near b

Local max : $f(b)$

Local min : $f(0) = 0$, $f(d)$

Abs max $f(a)$

Abs min $f(d)$